A New Algorithm to Simulate the First Exit Times of a Vector of Brownian Motions, with an Application to Finance

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SUMMARY

We provide a new methodology to simulate the first exit times of a vector of Brownian motions from an orthant. This new approach can be used to simulate the first exit times of dimension higher than two. When at least one Brownian motion has non-zero drift, the joint density function of the first exit times in N dimensions needs to be known, or approximated. However, when the drifts are all zero, a simpler simulation algorithm is obtained without using the joint density function.

Keywords and phrases: First exit times; Correlated Brownian motions; Gaussian copula; Multiple roots transformation

AMS Classification: 60G15; 60E05; 65C05; 65C30

1 Introduction

There are two main approaches to calculate the expected value of a function of joint first exit times of a process X from a sufficiently regular domain in $\mathbb{R}^N$, whether bounded or unbounded. The first approach is to set up the problem as a partial differential equation. When the process is Brownian motion, this equation is of either elliptic or parabolic type. The second approach is Monte Carlo simulation. The simplest implementation of Monte Carlo method is to discretize the stochastic differential equation of the underlying process, using for instance the Euler scheme, and to simulate the process until it reaches the barrier, i.e., the boundary of the domain (we will discuss and apply this method in Section 3.2 and Section 4). We refer to this indirect method as Euler-based Monte Carlo. According

to Fahim et al. [6], it is well known that numerical methods based on partial differential equations suffer from the curse of dimensionality: calculations and memory requirements typically increase exponentially as the dimension N increases. This is not the case for Monte Carlo simulation, where calculations are in general independent of the dimension of the problem.

A hybrid class of approaches uses Monte Carlo method to simulate the first exit times directly, i.e., without first simulating the underlying process X. There are two main advantages of this direct simulation approach, compared to Euler-based Monte Carlo. The first one relies on the fact that, in many applications, exit from a bounded domain is a rare event. It may take a very large amount of time (sometimes an infinite amount of time) for a process X to reach the boundary. The traditional solution to this problem is to use importance sampling. For advantages and pitfalls of importance sampling in financial engineering, we refer to [7] and the references therein. Another solution, in case the importance sampling density is difficult to construct, is to use interacting particle systems as in [4]. However, we can avoid this problem with direct simulation of the first exit times. The second advantage is its accuracy compared to Euler-based Monte Carlo method. Euler-based Monte Carlo tends to overestimate the first exit time: since the process is discretized, the method misses an excursion, i.e., an outcome where the process is in the domain at two contiguous epochs, but outside the domain at some time in-between these epochs. We note however that there are some techniques by Broadie et al. [3], Huh and Kolkiewicz [9] and Shevchenko [20] to alleviate this problem.

It is worth noting that direct first exit time Monte Carlo algorithms are not plagued by these two problems mentioned above but relying on the following two principles. First, the joint density of the first exit times must be either known or approximated correctly. This problem is often solved by a partial differential equation method, and is quite a difficult analytical problem in high dimensions if the different components of the process X are correlated. Secondly, exit times must be sampled from the joint distribution of the corresponding processes without, if possible, resorting to inversion and conditional Monte Carlo, which are very computationally intensive. For bounded domains, there is already rich literature. In [17] the bounded domain is approximated by a series of spheres; in [16] the bounded domain is approximated by a series of parallelepipeds. When the joint density of first exit times for general diffusions are unknown, the latter authors approximate the diffusion X by a Brownian motion on small bounded domains. We note that these methods provide better approximation than the simple Euler-based Monte Carlo method does, since the barrier does not need to be corrected.

Simulating the first exit time from an unbounded domain is a slightly different problem. We focus on a particular application in finance: the simulation of the joint default times of a set of N obligors. Black and Cox [2] were the first to model the default time of an obligor as the first time the value of the asset returns X crosses a lower, constant barrier. There is however no upper barrier on X. There have been many variations on this first exit time approach (which the finance literature refers to often as first passage time models). In [2],

X is a Brownian motion. Collin-Dufresne and Goldstein [5] use a more sophisticated model with mean-reversion in credit spreads. In the last ten years the interest has shifted, from examining a single obligor, to examining a large collection of obligors. Pools of large obligors were the building blocks of the credit default swaps which were heavily traded before the subprime crisis of 2008. It is now recognized that part of the blame of this crisis should be ascribed to the poor risk management of these contracts. Indeed, the task of estimating the parameters of these high-dimensional processes and of simulating them is still recognized as a difficult problem (see [1]). The difficulty of the simulation part of the problem does not come from the complexity of the unbounded domain (usually, a plain orthant), but the complexity of simulating in high dimensions. Zhou [22] provides a formula for the expected first default time (i.e., the expected value of the minimum of the first exit times), which is valid only when $N = 2$. Metzler [13] derives the formula for the joint density of the first exit times when $N = 2$, based on earlier work by Iyengar [10]. We generalize this formula to take into account the cases that were not given in [13]. Metzler [12, 13] also develops an algorithm to simulate the first exit times when the underlying two-dimensional Brownian motion has zero drift. We remind the readers that this is a more general problem than merely finding the expected value of the first default time, as the price of several complicated derivatives cannot be described as a function of the latter. Compared to Metzler [12, 13], our algorithm can be applied in the following cases:

1. $N = 2$ and at least one Brownian motion has non-zero drift.

2. $N > 2$ and all the Brownian motions are with zero drifts.

The main idea of generating the vector of first exit times in this paper is to transform the joint first exit times to some joint Chi-squared variables and then to generate the latter vector starting from standard Gaussian vector. From (2.3) below we see that, the transformations to Chi-squared distribution have single roots when the drift parameter $\mu_i = 0$ and multiple roots when $\mu_i \neq 0$. Hence our main results should be stated under two different scenarios: with drift and without drift. It is worth noting that, when the Brownian motions are correlated, our algorithm is not exact (except for two-dimensional Brownian motion with zero drift), however, statistical test and empirical study show that the error of simulation is acceptable in practice (see Sections 3.2 and 4).

The remainder of the paper proceeds as follows. In Section 2, we present our main theoretical results. The simulation of first exit times becomes the simulation of some 'basic' random variables. When the drift vector of X is non-zero the 'basic' random variables consist of uniform variables and Gaussian variables, and the simulation method depends on an explicit representation of the joint distribution of the first exit times, which is known only when $N = 2$. When the drifts are all zero, the method becomes simpler and much more powerful: the 'basic' random variables are Gaussian, and the explicit representation of the joint distribution of the first exit times is not needed. Section 3 summarizes the algorithms. In Section 4, we present an application of our methodology to the first exit time problem in finance. We compare our results to Zhou's analytical results [22] when $N = 2$, and to

Euler-based Monte Carlo simulation when $N \geq 2$.

2 Main Results

2.1 Statistical Modeling

For some integer $N \geq 2$, let us consider $\{(X_1(t), \ldots, X_N(t))\}_{t \geq 0}$, an N-dimensional Brownian motion, i.e. for $k \in \{1, \ldots, N\}$,

$$\mathrm{d}X_k(t) = \mu_k \, \mathrm{d}t + \sigma_k \, \mathrm{d}W_k(t), \tag{2.1}$$

where $\{W_k(t)\}_{t \geq 0}$'s are standard Brownian motions and $\mu_k \in \mathbb{R}, \sigma_k > 0$ are drift and volatility parameters respectively.

To introduce our estimation method let us start by some notation conventions:

- Denote by $(X_1(0), \ldots, X_N(0)) = (x_1, \ldots, x_N) \in \mathbb{R}^N$ the initial values.

- Let $(b_1, \ldots, b_N) \in \mathbb{R}^N$ be a vector of barriers. We assume that $b_k \neq x_k$ for all $k \in \{1, \ldots, N\}$.

- Let $(\tau_1, \ldots, \tau_N)$ be the vector of first exit times of $(X_1(t), \ldots, X_N(t))$ to the barriers $(b_1, \ldots, b_N)$. More precisely, for $k \in \{1, \ldots, N\}$,

$$\tau_k = \inf \big\{ t > 0 : X_k(t) = b_k, \ X_k(0) = x_k \big\}.$$

We are mainly interested in the problem of generating the joint first exit times $(\tau_1, \ldots, \tau_N)$. Let us recall that its marginal density is well known in the literature: let $\{X(t)\}_{t \geq 0}$ be a Brownian motion with drift $\mu \in \mathbb{R}$ and variance $Var(X(t)) = \sigma^2 t$ with $\sigma > 0$ and let $b \in \mathbb{R}$ be some given barrier and τ_b be the waiting time until the process $\{X(t)\}_{t \geq 0}$ first exits the horizontal line $Y = b$. Conditional on $X(0) = x \neq b$ and $\frac{\mu}{b-x} \geq 0$, τ_b has the probability density function (see for example [10]):

$$f_{\tau_b}(t) = \frac{|b - x|}{\sigma \sqrt{2\pi t^3}} \exp \Big(- \frac{(\mu t - (b - x))^2}{2\sigma^2 t} \Big). \tag{2.2}$$

The function f_{τ_b} is in fact an inverse Gaussian density. Remark that, if $\frac{\mu}{b-x} < 0$, this density of first exit time is defective, i.e., $\mathbb{P}(\tau_b < +\infty) < 1$. To overcome this inconvenience, we can generalize the density of first exit time as a mixture of distributions:

$$g_{\tau_b}(t) = f_{\tau_b}(t) \mathbb{1}_{[0,+\infty)}(t) + \Big(1 - \int_0^{+\infty} f_{\tau_b}(u) \, \mathrm{d}u \Big) \delta_{+\infty}(t),$$

where $\mathbb{1}_{[0,+\infty)}$ denotes the characteristic function and $\delta_{+\infty}$ is the Dirac measure.

A simple componentwise simulation could be used when the Brownian motions are independent. When the vector of Brownian motions consists of correlated components, the problem of simulating the joint first exit times could be rather challenging and we mainly focus on this problem.

2.2 First Exit Times of Brownian Motions with General Drifts and Multiple Roots Transformation

Generally, we suppose that $\{X_1(t),\ldots,X_N(t)\}$ are correlated drifted Brownian motions with $|Corr(X_i(t), X_j(t))| < 1$, for $i,j \in \{1,\ldots,N\}$ and $i \neq j$. Using the property of inverse Gaussian vector, we introduce a transformation of τ_i''s to Chi-squared random variables. More precisely, for $i \in \{1,\ldots,N\}$, define the transformation H_i as:

$$H_i(\tau_i) = \frac{(\mu_i \tau_i - (b_i - x_i))^2}{\sigma_i^2 \tau_i} \sim \chi^2(1), \tag{2.3}$$

where $\chi^2(1)$ denotes the Chi-squared distribution with one degree of freedom.

Now we are going to state the main results of this part. First we suppose μ takes any real value. The following proposition extends (2.3) to high dimension.

Proposition 2.1. *(See [21]) Let $(\tau_1,\ldots,\tau_N)$ be the vector of first exit times of an N-dimensional Brownian motions with parameters $(\mu_i, \sigma_i, x_i, b_i)$, $i \in \{1,\ldots,N\}$. Then there exists a vector of Chi-squared random variables with one degree of freedom $(\chi_1^2,\ldots,\chi_N^2)$ such that the following relation holds:*

$$\big(H_1(\tau_1),\ldots,H_N(\tau_N)\big) \sim \big(\chi_1^2,\ldots,\chi_N^2\big),$$

where $\{H_i\}_{i=1,\ldots,N}$ are given in (2.3).

Note that through the remaining part of this paper we suppose that for all $i \in \{1,\ldots,N\}$, $\frac{\mu_i}{b_i - x_i} \geq 0$, so that the density of τ_i is not defective. Now we explain our idea of simulation. By Proposition 2.1, we may consider using an inverse transform to simulate $(\tau_1,\ldots,\tau_N)$, however this inverse transform has multiple roots. Hence, by using a similar but more generalized method in [15], we are able to generate the first exit times approximately starting from the joint distribution of $(\chi_1^2,\ldots,\chi_N^2)$.

The following theorem outlines a simulation of the first exit times of correlated drifted Brownian motions.

Theorem 1. *Let $(\tau_1,\ldots,\tau_N)$ be the vector of first exit times given in Proposition 2.1 and denote by f its joint density. Let $(\chi_1^2,\ldots,\chi_N^2)$ be a vector of Chi-squared random variables with one degree of freedom satisfying*

$$\big(\chi_1^2,\ldots,\chi_N^2\big) \sim \big(H_1(\tau_1),\ldots,H_N(\tau_N)\big).$$

For $i = 1,\ldots,N$,

- *if $\mu_i \neq 0$, set*

$$\begin{aligned}
X_{i1} &= \frac{b_i - x_i}{\mu_i} + \frac{\sigma_i^2 \chi_i^2}{2\mu_i^2} - \frac{\sigma_i |\chi_i|}{2\mu_i^2}\sqrt{4\mu_i(b_i - x_i) + \sigma_i^2 \chi_i^2}; \\
X_{i2} &= \frac{(b_i - x_i)^2}{\mu_i^2 X_{i1}};
\end{aligned} \tag{2.4}$$

- *if $\mu_i = 0$, set*

$$X_{i1} = X_{i2} = \frac{(b_i - x_i)^2}{\sigma_i^2 \chi_i^2}. \tag{2.5}$$

For $k \in \{1, \dots, N\}$ and $u_k \in \{1, 2\}$, define

$$S_u = \{1, 2\}^N \setminus \{(u_1, \dots, u_N)\}$$

and

$$p_{u_1 u_2 \cdots u_N}$$
$$= \left(1 + \sum_{(i_1, \dots, i_N) \in S_u} \left(\prod_{k=1}^N \left|\frac{H_k'(X_{ku_k})}{H_k'(X_{ki_k})}\right|\right) \frac{f(X_{1i_1}, X_{2i_2}, \dots, X_{Ni_N})}{f(X_{1u_1}, X_{2u_2}, \dots, X_{Nu_N})}\right)^{-1},$$

where H_k' denotes the derivative of H_k. Denote by λ the Lebesgue measure on $\mathbb{R}$. Let $\{I_{u_1 \cdots u_N}\}_{u_1, \dots, u_N \in \{1, 2\}}$ be any partition of interval $[0, 1]$ satisfying

$$\mathbb{P}\big(\lambda(I_{u_1 \cdots u_N}) = p_{u_1 \cdots u_N}\big) = 1, \tag{2.6}$$

then we have

$$(\tau_1, \dots, \tau_N) \sim \sum_{u_1, \dots, u_N \in \{1, 2\}} \big(X_{1u_1}, \dots, X_{Nu_N}\big) \mathbb{1}_{I_{u_1 \cdots u_N}}(U),$$

where $U \sim Unif(0, 1)$ is independent of $(\chi_1^2, \dots, \chi_N^2)$.

It is very difficult to generate the random vector $(\chi_1^2, \dots, \chi_N^2)$ by its exact distribution since: the vector $(\chi_1^2, \dots, \chi_N^2)$ can not be exactly generated from a Gaussian copula. To show this fact we suppose $N = 2$ and let (Z_1, Z_2) be a Gaussian vector verifying $Z_1^2 = \chi_1^2$, $Z_2^2 = \chi_2^2$, then we should have (see [19], Lemma 5.3.4)

$$Cov(\chi_1^2, \chi_2^2) = Cov(Z_1^2, Z_2^2) = 2\big(Cov(Z_1, Z_2)\big)^2 \geq 0.$$

But numerical integration (for instance by using MATLAB) shows $Cov(\chi_1^2, \chi_2^2) < 0$ for some values of $(x_1, b_1, \sigma_1, x_2, b_2, \sigma_2, \rho)$. For example, taking $b_1 - x_1 = b_2 - x_2 = -5, \sigma_1 = \sigma_2 = 1, \rho = -0.5$ leads to $Cov(\chi_1^2, \chi_2^2) = -0.4007 < 0$. Although it is difficult to actually generate two dependent first exit times when they have distributions that are not from a standard multivariate distribution, however, we can expect to simulate data following the same marginal distribution and covariance structure via $(\tau_1, \dots, \tau_N)$. More precisely, we introduce the following approximation:

Definition 2.1. Two second order (i.e., with finite covariance matrix) random vectors $(X_1, \dots, X_N)$ and $(Y_1, \dots, Y_N)$ are said to be approximately identically distributed if for $i \in \{1, \dots, N\}$, their marginal distributions are equal: $X_i \sim Y_i$; and their covariance matrices are identical:

$$Cov\big((X_1, \dots, X_N)\big) = Cov\big((Y_1, \dots, Y_N)\big).$$

We denote this relation by $(X_1, \dots, X_N) \overset{approx}{\sim} (Y_1, \dots, Y_N)$.

Note that nowadays in financial risk aggregation, elliptical copulas (including Gaussian copula) are widely used to simulate dependent data, since they provide a way to model correlated multivariate variables. In our case, we choose the Gaussian copula to generate first exit times following their exact means and covariance matrix. This is the so-called NORTA (normal-to-anything) method. Here we recall a nice result on the simulation of joint uniform random variables by following their covariance structure, starting from Gaussian vector:

Lemma 2.1. *Assume that $U_1, U_2 \sim Unif(0,1)$ have correlation ρ_U, then there exist $Z_1, Z_2 \sim \mathcal{N}(0,1)$ with correlation ρ_Z satisfying $\rho_Z = 2\sin\left(\frac{\pi}{6}\rho_U\right)$, and*

$$(U_1, U_2) \overset{approx}{\sim} (\varphi(Z_1), \varphi(Z_2)),$$

where φ denotes the cumulative distribution function of the standard normal distribution.

This result is known as the exact relationship between Spearman correlation and Bravais-Pearson correlation (see for example [8]). The approximation given in Definition 2.1 and Lemma 2.1, allow us to simulate the first exit times in the following two situations:

Case 1: When $N = 2$, and the Brownian motions have non-zero drifts.

In this case the transformations $\{H_k\}_{k=1,2}$ have multiple roots and the explicit formula of f, the joint density of (τ_1, τ_2) can be derived. Hence one can determine the partitions $\{I_{ij}\}_{i,j=1,2}$ given in (2.6) explicitly. The joint density f of (τ_1, τ_2) is given in the following:

Proposition 2.2. *Let $(X_1(t), X_2(t))$ be a two-dimensional Brownian motion respectively with drifts $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1^2, \sigma_2^2 > 0$. Then starting from $(x_1, x_2) \in \mathbb{R}^2$, the joint density of the first exit times (τ_1, τ_2) to the barriers $(b_1, b_2) \in \mathbb{R}^2$ is given as*

1. *Let $(\gamma_1, \gamma_2) = \left(\frac{\sigma_2\mu_1 - \sigma_1\mu_2\rho}{\sigma_1\sigma_2\sqrt{1-\rho^2}}, \frac{\mu_2}{\sigma_2}\right)$, and $(\tilde{\mu}_1, \tilde{\mu}_2) = \left((sgn(x_1 - b_1))\gamma_1, (sgn(x_2 - b_2))\gamma_2\right)$. For $0 < s < t$,*

$$f(s,t) = \sqrt{\frac{\pi}{2}} \frac{\sin\alpha}{\alpha^2 s \sqrt{(t-s)^3}} e^{-r_0(\frac{r_0}{2s} + \tilde{\mu}_1\cos\theta_0 + \tilde{\mu}_2\sin\theta_0) - \frac{\tilde{\mu}_1^2 s + \tilde{\mu}_2^2 t}{2}}$$

$$\times \sum_{n=1}^{+\infty} n\sin\left(\frac{n\pi(\alpha - \theta_0)}{\alpha}\right) \int_0^{+\infty} e^{r\tilde{\mu}_1\cos\alpha - r^2(\frac{t - s\cos^2\alpha}{2s(t-s)})} I_{n\pi/\alpha}\left(\frac{rr_0}{s}\right) \mathrm{d}r.$$

$$(2.7)$$

2. For $0 < t < s$,

$$f(s,t) = \sqrt{\frac{\pi}{2}} \frac{\sin \alpha}{\alpha^2 t \sqrt{(s-t)^3}} \exp\left(-r_0\left(\frac{r_0}{2t} + \tilde{\mu}_1 \cos \theta_0 + \tilde{\mu}_2 \sin \theta_0\right)\right)$$

$$-\frac{(\tilde{\mu}_1^2 + \tilde{\mu}_2^2)t}{2} - \frac{(\tilde{\mu}_1 \sin \alpha - \tilde{\mu}_2 \cos \alpha)^2 (s-t)}{2}\right) \sum_{n=1}^{+\infty} n \sin\left(\frac{n\pi\theta_0}{\alpha}\right)$$

$$\times \int_0^{+\infty} e^{-r(\tilde{\mu}_1 \cos^2 \alpha + \tilde{\mu}_2 \sin \alpha \cos \alpha) - r^2 \left(\frac{s-t \cos^2 \alpha}{2t(s-t)}\right)} I_{n\pi/\alpha}\left(\frac{rr_0}{t}\right) \mathrm{d}r,$$

$$(2.8)$$

where

$$\tilde{\rho} = \left(sgn\left(\frac{b_1 - x_1}{b_2 - x_2}\right)\right)\rho \ \text{ with } sgn(\cdot) \text{ being the sign function,}$$

$$\alpha = \begin{cases} \pi + \tan^{-1}\left(-\frac{\sqrt{1-\tilde{\rho}^2}}{\tilde{\rho}}\right) & \text{if } \tilde{\rho} > 0, \\ \frac{\pi}{2} & \text{if } \tilde{\rho} = 0, \\ \tan^{-1}\left(-\frac{\sqrt{1-\tilde{\rho}^2}}{\tilde{\rho}}\right) & \text{if } \tilde{\rho} < 0, \end{cases}$$

$$r_0 = \frac{1}{\sigma_1 \sigma_2} \sqrt{\frac{(b_1 - x_1)^2 \sigma_2^2 + (b_2 - x_2)^2 \sigma_1^2 - 2|(b_1 - x_1)(b_2 - x_2)|\tilde{\rho}\sigma_1\sigma_2}{1 - \tilde{\rho}^2}},$$

$$\theta_0 = \begin{cases} \pi + \tan^{-1}\left(\frac{\sigma_1|b_2-x_2|\sqrt{1-\tilde{\rho}^2}}{|b_1-x_1|\sigma_2-\tilde{\rho}|b_2-x_2|\sigma_1}\right) & \text{if } |b_1 - x_1|\sigma_2 < \tilde{\rho}|b_2 - x_2|\sigma_1, \\ \frac{\pi}{2} & \text{if } |b_1 - x_1|\sigma_2 = \tilde{\rho}|b_2 - x_2|\sigma_1, \\ \tan^{-1}\left(\frac{\sigma_1|b_2-x_2|\sqrt{1-\tilde{\rho}^2}}{|b_1-x_1|\sigma_2-\tilde{\rho}|b_2-x_2|\sigma_1}\right) & \text{if } |b_1 - x_1|\sigma_2 > \tilde{\rho}|b_2 - x_2|\sigma_1, \end{cases}$$

and I_β with $\beta > 0$ denotes the modified Bessel function of the first kind of order β.

Inspired by Lemma 2.1, we provide an approximation of the distribution of (τ_1, τ_2) starting from a standard Gaussian vector (Z_1, Z_2):

Proposition 2.3. *Let (τ_1, τ_2) be the first exit times of $(X_1(t), X_2(t))$ with correlation $\rho \in (-1, 1)$. Set (Z_1, Z_2) with standard normal components satisfying:*

$$Corr(Z_1, Z_2)$$
$$= 2 \sin\left(\frac{\pi}{6} Corr\left(2\varphi\left(\frac{|\mu_1\tau_1 - (b_1 - x_1)|}{\sigma_1\sqrt{\tau_1}}\right) - 1, 2\varphi\left(\frac{|\mu_2\tau_2 - (b_2 - x_2)|}{\sigma_2\sqrt{\tau_2}}\right) - 1\right)\right).$$

$$(2.9)$$

Let

$$(\chi_1^2, \chi_2^2) = \left(\left(\varphi^{-1}\left(\frac{\varphi(Z_1)+1}{2}\right)\right)^2, \left(\varphi^{-1}\left(\frac{\varphi(Z_2)+1}{2}\right)\right)^2\right).$$

$$(2.10)$$

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Let $U \sim Unif(0,1)$ be independent of χ_1^2, χ_2^2. Then, the distribution of (τ_1, τ_2) can be approximated by that of

$$\sum_{i,j \in \{1,2\}} (X_{1i}, X_{2j}) \mathbb{1}_{I_{ij}}(U), \tag{2.11}$$

in the sense that

$$(\chi_1^2, \chi_2^2) \overset{approx}{\sim} (H_1(\tau_1), H_2(\tau_2)),$$

where for $i, j \in \{1, 2\}$, (X_{1i}, X_{2j}) is defined in (2.4), and I_{ij} is given in (2.6).

This proposition is a straightforward consequence of Lemma 2.1 and Theorem 1 by taking $N = 2$. Hence we omit its proof. Note that in (2.9) the correlation

$$Corr\Big(2\varphi\Big(\frac{|\mu_1\tau_1 - (b_1 - x_1)|}{\sigma_1\sqrt{\tau_1}}\Big) - 1, 2\varphi\Big(\frac{|\mu_2\tau_2 - (b_2 - x_2)|}{\sigma_2\sqrt{\tau_2}}\Big) - 1\Big)$$

can be computed numerically by using the joint density of (τ_1, τ_2). Although calculating the correlation by numerical integration in two dimensions is as difficult as calculating the mean exit time, we emphasize that this calculation can be done offline, while exit time simulation usually is not. For instance, in Section 4, one is given a single model of joint asset returns, for which the correlation $Corr(Z_1, Z_2)$ must be calculated once. However, several contingent claims can be written on these assets. The valuation of each of them requires an independent simulation, and every simulation shares the same value of $Corr(Z_1, Z_2)$, whose calculation does not need to be repeated. More importantly, as we will show in the next subsection, when $N > 2$ only pairwise correlations need to be numerically calculated.

Case 2: When $N \geq 2$, and the Brownian motions are without drifts.

In this case the joint density of $(\tau_1, \ldots, \tau_N)$ is generally unknown (or too complicated to get). We remark that it is not yet tractable in the literature to exactly simulate $(\tau_1, \ldots, \tau_N)$ when its joint density is unknown. Fortunately, when all the drifts are vanishing, the transformations H_i's have single roots, and as a consequence we have avoided the problem of selection among multiple roots of transform using the joint density of first exit times. An example of high dimensional simulation is given in Section 4.2. We are going to show the generating method for this case as well as the algorithms in the following sections.

2.3   First Exit Times of Brownian Motions with Zero Drift

When $(\mu_1, \mu_2) = 0$, the reduced joint density of $f(s,t)$ in Proposition 2.2 and the exact simulation of (τ_1, τ_2) are given in the following Theorem 2 (i). From Theorem 2 (ii), we see $\{H_i\}_{i=1,2}$ reduce to a single root transformations.

Theorem 2. (i) *The random vector (τ_1, τ_2) has the following joint density:*

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1. *For $0 < s < t$,*

$$f(s,t) = \frac{\pi \sin \alpha}{2\alpha^2 \sqrt{s(t - s \cos^2 \alpha)(t - s)}} e^{-\frac{r_0^2(t - s \cos 2\alpha)}{2s((t-s)+(t-s \cos 2\alpha))}}$$

$$\times \sum_{n=1}^{\infty} n \sin\left(\frac{n\pi(\alpha - \theta_0)}{\alpha}\right) I_{n\pi/2\alpha}\left(\frac{r_0^2(t - s)}{2s((t-s) + (t - s \cos 2\alpha))}\right).$$

2. *For $0 < t < s$,*

$$f(s,t) = \frac{\pi \sin \alpha}{2\alpha^2 \sqrt{t(s - t \cos^2 \alpha)(s - t)}} e^{-\frac{r_0^2(s - t \cos 2\alpha)}{2t((s-t)+(s-t \cos 2\alpha))}}$$

$$\times \sum_{n=1}^{\infty} n \sin\left(\frac{n\pi\theta_0}{\alpha}\right) I_{n\pi/2\alpha}\left(\frac{r_0^2(s - t)}{2t((s-t) + (s - t \cos 2\alpha))}\right).$$

(ii) *There exist two Chi-squared random variables χ_1^2, χ_2^2 verifying*

$$\mathbb{E}\left(\chi_1^2\chi_2^2\right) = \frac{(b_1 - x_1)^2(b_2 - x_2)^2}{(\sigma_1\sigma_2)^2} \int_0^{+\infty} \int_0^{+\infty} \frac{f(s,t)}{st} \, ds \, dt,$$

such that, with probability 1, $(\tau_1, \tau_2) = \left(\frac{(b_1-x_1)^2}{\sigma_1^2\chi_1^2}, \frac{(b_2-x_2)^2}{\sigma_2^2\chi_2^2}\right)$.

Theorem 2 is a straightforward consequence of Proposition 2.2 and Proposition 2.1, where one takes $\mu_1 = \mu_2 = 0$. Remark that Metzler [13] presented the joint density of (τ_1, τ_2) only in the case when $x_i > b_i = 0$, $i = 1, 2$. Here we have generalized his result to any real values of (x_i, b_i). Moreover, this theorem could provide a simple and fast simulation of the first exit times. We refer to the following corollary:

Corollary 2.1. There exists a Gaussian vector (Z_1, Z_2) with $Z_1, Z_2 \sim \mathcal{N}(0, 1)$ and

$$Corr(Z_1, Z_2) = 2 \sin\left(\frac{\pi}{6} Corr\left(2\varphi\left(\frac{|b_1 - x_1|}{\sigma_1\sqrt{\tau_1}}\right) - 1, 2\varphi\left(\frac{|b_2 - x_2|}{\sigma_2\sqrt{\tau_2}}\right) - 1\right)\right),$$

such that the distribution of (τ_1, τ_2) can be approximated by that of

$$\left(\frac{(b_1 - x_1)^2}{\sigma_1^2(\varphi^{-1}(\frac{\varphi(Z_1)+1}{2}))^2}, \frac{(b_2 - x_2)^2}{\sigma_2^2(\varphi^{-1}(\frac{\varphi(Z_2)+1}{2}))^2}\right)$$

in the sense that

$$\left((\varphi^{-1}(\frac{\varphi(Z_1)+1}{2}))^2, (\varphi^{-1}(\frac{\varphi(Z_2)+1}{2}))^2\right) \overset{approx}{\sim} \left(\frac{(b_1 - x_1)^2}{\sigma_1^2\tau_1}, \frac{(b_2 - x_2)^2}{\sigma_2^2\tau_2}\right).$$

By using the property that the zero-mean Gaussian vector's distribution is determined only by its covariance matrix, the result of Corollary 2.1 can be straightforwardly extended to high-dimensional correlated Brownian motions with zero drift.

Corollary 2.2. Let the integer $N \geq 2$, let $(Z_1, \ldots, Z_N) \sim \mathcal{N}(0, \Sigma)$ be any Gaussian vector satisfying,

$$\Sigma = (r_{ij})_{N \times N}, \text{ with } r_{ii} = 1 \text{ for } i \in \{1, 2, \ldots, N\}$$

and for different $i, j \in \{1, 2, \ldots, N\}$,

$$r_{ij} = 2 \sin \left(\frac{\pi}{6} Corr \left(2\varphi \left(\frac{|b_i - x_i|}{\sigma_i \sqrt{\tau_i}} \right) - 1, 2\varphi \left(\frac{|b_j - x_j|}{\sigma_j \sqrt{\tau_j}} \right) - 1 \right) \right), \tag{2.12}$$

where each pair (τ_i, τ_j) is with parameters $(x_i, b_i, \sigma_i, x_j, b_j, \sigma_j, \rho_{ij})$. Then the random vector $(\tau_1, \ldots, \tau_N)$ can be approximated by

$$\left(\frac{(b_1 - x_1)^2}{\sigma_1^2 (\varphi^{-1}(\frac{\varphi(Z_1)+1}{2}))^2}, \ldots, \frac{(b_N - x_N)^2}{\sigma_N^2 (\varphi^{-1}(\frac{\varphi(Z_N)+1}{2}))^2} \right) \tag{2.13}$$

in the sense that

$$\left((\varphi^{-1}(\frac{\varphi(Z_i) + 1}{2}))^2 \right)_{i=1,\ldots,N} \overset{approx}{\sim} \left(\frac{(b_i - x_i)^2}{\sigma_i^2 \tau_i} \right)_{i=1,\ldots,N}.$$

3 Algorithms and Statistical Tests

3.1 Algorithms of First Exit Times

Now we introduce a simulation algorithm of first exit times for Brownian motions. We are mainly interested in sampling the first exit times from correlated Brownian motions. The algorithms are given as follows:

Case 1: Two-dimensional Brownian Motions with Drifts.

For a general two-dimensional vector of Brownian motions, the algorithm can be provided based on Proposition 2.3.

1. Generate a Gaussian vector (Z_1, Z_2) satisfying (2.9).

2. For $i = 1, 2$, determine (X_{i1}, X_{i2}) by using (2.4) and (2.5).

3. Determine (χ_1^2, χ_2^2) according to (2.10).

4. Generate (τ_1, τ_2) by using (2.11).

Case 2: N-dimensional Brownian Motions without Drifts.

When the drift is zero, we supply an algorithm to simulate $(\tau_1, \ldots, \tau_N)$ based on Corollary 2.2.

1. Calculate $Corr\left(2\varphi\left(\frac{|b_i-x_i|}{\sigma_i\sqrt{\tau_i}}\right)-1, 2\varphi\left(\frac{|b_j-x_j|}{\sigma_j\sqrt{\tau_j}}\right)-1\right)$ for $i,j \in \{1,\ldots,N\}$, $i \neq j$ by using f_{ij}, the joint density of (τ_i, τ_j):

$$Corr\left(2\varphi\left(\frac{|b_i-x_i|}{\sigma_i\sqrt{\tau_i}}\right)-1, 2\varphi\left(\frac{|b_j-x_j|}{\sigma_j\sqrt{\tau_j}}\right)-1\right)$$
$$= 12\left(\int_{\mathbb{R}_+^2}\left(2\varphi\left(\frac{|b_i-x_i|}{\sigma_i\sqrt{s}}\right)-1\right)\left(2\varphi\left(\frac{|b_j-x_j|}{\sigma_j\sqrt{t}}\right)-1\right)f_{ij}(s,t)\,\mathrm{d}s\,\mathrm{d}t - \frac{1}{4}\right).$$

2. Determine r_{ij} for $i,j \in \{1,\ldots,N\}$, $i \neq j$, by using Equation (2.12).

3. Generate $(Z_1,\ldots,Z_N) \sim \mathcal{N}(0,\Sigma)$, with $\Sigma = (r_{ij})_{N \times N}$.

4. Generate $(\tau_1,\ldots,\tau_N)$ by using (2.13).

We emphasize that only pairwise correlations need to be numerically calculated. Thus, one of the main advantages of our algorithm is its little numerical complexity: rather than performing an N-dimensional integration the algorithm performs a calculation of two-dimensional integrations.

3.2 A Statistical Test of the Algorithms

Although the simulation of the joint first exit times is approximate, it has been shown that the true distribution of the joint first times is quite similar to the one generated by using Gaussian copula. For example, Overbeck and Schmidt [18] compared a Gaussian copula model to a calibration time changed model using the fair basket default swap spreads data and showed that the results are quite close. From their numerical results, "the seed variance of fair the first-to-default spread is less than 0.10% and much smaller for the other spreads"; McLeish [14] established an estimator of the maxima of two correlated Brownian motions using some normal vector, and he showed that the approximation is remarkably accurate. In addition, Metzler [12] (Pages 50-51) has compared the true distribution of the first exit time of a 3-dimensional correlated Brownian motions (in this example the barriers are supposed to be equal) to the one from its corresponding Gaussian copula, and concludes that the first exit time in a "correlated Brownian drivers" model is quite similar to the one in a Gaussian copula model. Here we use another statistical approach to compare our method to Monte Carlo's method. More comparisons using numerical results are made in the next section. Recall that the Kolmogorov-Smirnov test (K-S test) is used to check the equality of probability distributions. In order to compare random vectors we take a high dimensional K-S test (see for instance [11]), programmed in MATLAB. Let (τ_1, τ_2) be the random vector generated by using Monte Carlo algorithm and $(\widetilde{\tau}_1, \widetilde{\tau}_2)$ be the one generated by using transformation with multiple roots.

We set the parameters for $i = 1, 2$, $(x_i, b_i, \sigma_i) = (\log(5), 0, 1)$ and the test's significant level $\alpha = 0.01$. 10^5 realizations are generated by using each method. Now we set up the hypothesis:

$$\mathcal{H}_0 : (\tau_1, \tau_2) \sim (\widetilde{\tau}_1, \widetilde{\tau}_2) \text{ and } \mathcal{H}_1 : (\tau_1, \tau_2) \nsim (\widetilde{\tau}_1, \widetilde{\tau}_2).$$

The results $\mathcal{H}_0$ presented in the following Table 1 show that we won't reject the fact that $(\tau_1, \tau_2) \sim (\widetilde{\tau}_1, \widetilde{\tau}_2)$ with confidence 0.99 when $N = 2$. However, the dispersion could be increasing when the dimension N increases.

Parameters	K-S Test Result by MATLAB
$\mu_1 = \mu_2 = 0,\ \rho = 0.1$	$\mathcal{H}_0$
$\mu_1 = \mu_2 = -0.05,\ \rho = 0.1$	$\mathcal{H}_0$
$\mu_1 = \mu_2 = -0.05,\ \rho = 0.5$	$\mathcal{H}_0$

Table 1: K-S test with parameters $(x_i, b_i, \sigma_i) = (\log(5), 0, 1)$, $\alpha = 0.01$

4 Application to a Multiple-factor Model of Portfolio Default

The simulation of first exit times of Brownian motions has its interests in many credit risk applications. Let X_i be the logarithm of the total value of the assets of firm i, where $i = 1, ..., N$. Assume that X_i satisfies the stochastic differential equation (2.1). For $i \neq j$, we denote by ρ_{ij} the correlation per unit of time between X_i and X_j.

The substantial decrease of a firm's asset value is the main reason of the default of the firm. Black and Cox [2] defined the threshold value as the minimum asset value of the firm required by the debt covenants. If X_i falls to the threshold value b_i, the bond holder are entitled to a 'deficiency claim' which can force the firm into bankruptcy. Zhou [22] defined the barrier b_i as the logarithm of the sum of the short-term debt principal and one half of long-term debt principal of a firm. x_i is the logarithm of the current total asset values of firms. With input parameters $(\mu_i, \sigma_i, x_i, b_i, \rho_{ij})$, we denote the output by P_i, the probability that i firms out of N in a portfolio have defaulted by a time horizon T. For simplicity, the subsequent numerical simulations are based on the assumption that the underlying Brownian motions have the same drift terms and standard deviations. Since, for financial products, most correlations lie between -0.5 and 0.5, three different levels of correlations (i.e., we denote by $\rho = \rho_{ij} = -0.5, 0.1, 0.5$) are examined in the following tables. Zhou [22]'s parameters are applied here: $(\sigma_i, x_i, b_i) = (1, \log(5), 0)$. Over a short horizon, quick default events are rare and the probabilities of multiple defaults could be extremely small, hence a long horizon $T = 10$ (years) is used to overcome this problem. We present the multiple default probabilities using two different methods: our method and the Euler-based Monte Carlo algorithm. We compare them to the method in [22] (although our method is more general than Zhou's result, we compare our method to his for accuracy), which is exact. In the following tables, the step size of the Euler-based Monte Carlo method is equal to 0.0015625 and the total number of scenarios is 10^6. Note that the Monte Carlo method is always an alternative, however it becomes very computationally intensive when a large number of scenarios need to be performed.

4.1 Simulations of First Exit Times of Two-dimensional Correlated Brownian Motions

The examples without drift terms and with drift terms are both examined in the following Tables 2-7. Three different level correlations are presented to demonstrate how the correlation of logarithm of asset value affects the default probability distribution. In Table 2, $\rho = 0.1$ means the movements of two firms' logarithm of asset value are weakly correlated with each other. The fact that $\rho = 0.5$ shows these movements are strongly positively correlated. The simulation results illustrate the fact that the stronger are the assets' correlations, the higher are the probabilities of multiple defaults. Zhou [22] indicated that the default correlation and the asset level correlation have the same sign which explains our results here. For instance, a drop in one firm's asset value leads to a decrease in another firm's asset value (which is closer to the default boundary), and then leads to a rise of probability of both firms' defaults. If two firms' asset values do not move towards the same direction, this behavior leads to a drop in likelihood of multiple default events.

Compared to the analytical results in [22], our method is not completely unbiased, but the results are very promising when $\rho > 0$. In fact from Tables 2-5, the estimated value of our method has an error rate less than 3.57%. When $\rho < 0$, our numerical results (Tables 6-7) show that the error rate is less than 20%. However, in practice we rarely consider the case $\rho < 0$, since it is quite artificial (see [18]).

	Our Method	Zhou (2001)	Euler-based
P_2	0.390521	0.386337	0.380781
P_1	0.440707	0.448901	0.451572
P_0	0.168782	0.164761	0.167657

Table 2: Two-dimensional simulation results with $\rho = 0.1, \mu_1 = \mu_2 = 0$

	Our Method	Zhou (2001)	Euler-based
P_2	0.445721	0.446907	0.440361
P_1	0.426332	0.424764	0.428239
P_0	0.127957	0.128328	0.131400

Table 3: Two-dimensional simulation results with $\rho = 0.1, \mu_1 = \mu_2 = -0.05$

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	Our Method	Zhou (2001)	Euler-based
P_2	0.439642	0.445308	0.440013
P_1	0.344621	0.330958	0.331241
P_0	0.215747	0.223732	0.228756

Table 4: Two-dimensional simulation results with $\rho = 0.5, \mu_1 = \mu_2 = 0$

	Our Method	Zhou (2001)	Euler-based
P_2	0.505348	0.502006	0.498137
P_1	0.30397	0.314566	0.315721
P_0	0.190682	0.183426	0.186142

Table 5: Two-dimensional simulation results with $\rho = 0.5, \mu_1 = \mu_2 = -0.05$

	Our Method	Zhou (2001)	Euler-based
P_2	0.325874	0.308726	0.301252
P_1	0.570430	0.604123	0.607972
P_0	0.103696	0.087150	0.090786

Table 6: Two-dimensional simulation results with $\rho = -0.5, \mu_1 = \mu_2 = 0$

	Our Method	Zhou (2001)	Euler-based
P_2	0.372292	0.376896	0.367961
P_1	0.566252	0.564787	0.571982
P_0	0.061456	0.058316	0.060057

Table 7: Two-dimensional simulation results with $\rho = -0.5, \mu_1 = \mu_2 = -0.05$

4.2 First Exit Times Simulations for High-dimensional Correlated Brownian Motions

We note that our method is roughly as good as Euler-based Monte Carlo method. This is important, as the method in [22] is not applicable in higher dimensions. Some simulation results of first exit times for high-dimensional Brownian motions are listed in Table 8. The subsequent simulations are based on the simple examples without drift terms. Since Zhou's

results can not work on higher dimensional cases, only Monte Carlo method is used here. For simplicity, all the correlations are set to be 0.1. For instance, the correlation matrix $Corr(X_1, X_2, X_3)$ of three firms' logarithm of asset value X_i where $i = 1, 2, 3$ in Table 8 is

$$Corr(X_1, X_2, X_3) \quad = \quad \begin{pmatrix} 1 & 0.1 & 0.1 \\ 0.1 & 1 & 0.1 \\ 0.1 & 0.1 & 1 \end{pmatrix}.$$

The simulation results are consistent with the fact that the Monte Carlo method tends to under-estimate the multiple defaults probabilities. According to [4], when the number of firms increases, this estimation could have larger deviation. However, compared to the results in [22] which can not calculate multiple default probabilities for more than two firms, the new proposed method can be advantageously expanded to high dimensional Brownian motions. In Table 8, the error rate of our estimation compared to Monte Carlo algorithm is less than 5.38%, which is acceptable in practice.

	Our Method	Euler-based
P_3	0.257971	0.250633
P_2	0.395472	0.403186
P_1	0.267637	0.271008
P_0	0.078920	0.075173

Table 8: Three-dimensional simulation results with $Corr(X_1, X_2, X_3)$

5   Conclusion

A new numerical algorithm is proposed to solve the first exit times of a vector of Brownian motions with zero and non-zero drifts. Compared to the methods in [12, 13] and Zhou's analytical and numerical results, the advantages of our method are the following: When $N = 2$, our algorithms can take into account drifts and can be easily extended to high dimensions, especially when the drifts are zero. As a consequence, our method allows calculating the expected value of more general functions of the first exit times, not constrained to the minimum of the latter, which is the traditional advantage of simulation.

6   Appendix

6.1   Proof of Theorem 1.

Let $(\tau_1, \tau_2, \ldots, \tau_N)$ be a vector of first exit times with parameters $\left\{ (\mu_i, \sigma_i, x_i, b_i) \right\}_{i=1,\ldots,N}$. We assume that $x_i \neq b_i$ and $\frac{\mu_i}{b_i - x_i} \geq 0$ for $i \in \{1, \ldots, N\}$ and the joint density of $(\tau_1, \ldots, \tau_N)$

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is f. Hence, still by [21], $(\tau_1, \ldots, \tau_N)$ is the solution of the following equation:

$$\big(H_1(\tau_1), \ldots, H_N(\tau_N)\big) = \big(\chi_1^2, \ldots, \chi_N^2\big). \tag{6.1}$$

Notice that for $i = 1, \ldots, N$, $v_i \geq 0$, the equation

$$\big(H_1(x_1), \ldots, H_N(x_n)\big) = (v_1, \ldots, v_N)$$

has 2^N roots (can be surplus) $\big\{ \big(x_{1u_1}, x_{2u_2}, \ldots, x_{Nu_N}\big) : (u_1, \ldots, u_N) \in \{1,2\}^N \big\}$,

- if $\mu_i \neq 0$, set

$$
\begin{aligned}
x_{i1} &= \frac{b_i - x_i}{\mu_i} + \frac{\sigma_i^2 v_i}{2\mu_i^2} - \frac{\sigma_i}{2\mu_i^2}\sqrt{4\mu_i(b_i - x_i)v_i + \sigma_i^2 v_i^2}; \\
x_{i2} &= \frac{(b_i - x_i)^2}{\mu_i^2 x_{i1}};
\end{aligned}
$$

- if $\mu_i = 0$, set

$$x_{i1} = x_{i2} = \frac{(b_i - x_i)^2}{\sigma_i^2 v_i}. \tag{6.2}$$

The remaining problem is choosing one among the 2^N roots according to the observed value of the Chi-squared vector $\big(\chi_1^2, \ldots, \chi_N^2\big)$. We explain how this process can be taken. For a particular root $\big(x_{1u_1}, x_{2u_2}, \ldots, x_{Nu_N}\big)$, we compute the probability of choosing it. Let $\epsilon > 0$ be arbitrarily small and let $H_i^{-1}(v_i - \epsilon, v_i + \epsilon)$ be the inverse image of $(v_i - \epsilon, v_i + \epsilon)$. For $i = 1, \ldots, N$, let the intervals $(y_{iu_i}^{(1)}, y_{iu_i}^{(2)}) \subset H_i^{-1}(v_i - \epsilon, v_i + \epsilon)$ be disjoint for each $u_i \in \{1, 2\}$ and contain the u_i-th root of equation $H_i(x) = v_i$: $x_{iu_i} \in (y_{iu_i}^{(1)}, y_{iu_i}^{(2)})$. Notice that we should choose $\big(x_{1u_1}, x_{2u_2}, \ldots, x_{Nu_N}\big)$ as the observation of $(\tau_1, \ldots, \tau_N)$ if $\tau_i \in (y_{iu_i}^{(1)}, y_{iu_i}^{(2)})$ for $i = 1, \ldots, N$. Therefore the probability of choosing the particular root $\big(x_{1u_1}, \ldots, x_{Nu_N}\big)$ is given as

$$
\begin{aligned}
&\mathbb{P}^\epsilon\big((x_{1u_1}, \ldots, x_{Nu_N})\big) \\
&= \frac{\mathbb{P}\big(\tau_1 \in (y_{1u_1}^{(1)}, y_{1u_1}^{(2)}), \ldots, \tau_N \in (y_{Nu_N}^{(1)}, y_{Nu_N}^{(2)})\big)}{\sum_{j_1, \ldots, j_N \in \{1,2\}} \mathbb{P}\big(\tau_i \in (y_{1j_1}^{(1)}, y_{1j_1}^{(2)}), \ldots, \tau_N \in (y_{Nj_N}^{(1)}, y_{Nj_N}^{(2)})\big)}.
\end{aligned}
$$

Observe that $(y_{iu_i}^{(1)}, y_{iu_i}^{(2)}) \to x_{iu_i}$ as $\epsilon \to 0^+$. Hence letting $\mathbb{P}^\epsilon$ denote the probability measure which depends on ϵ and letting $\epsilon \to 0^+$, by a similar principle to L'Hôpital's rule (see

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Equation (3) in [15]), we get

$$\mathbb{P}_{(u_1,u_2,\ldots,u_N)}((v_1,\ldots,v_N)) := \lim_{\epsilon \to 0^+} \mathbb{P}^\epsilon\big((x_{1u_1},\ldots,x_{Nu_N})\big)$$

$$= \lim_{\epsilon \to 0^+} \left\{ 1 + \sum_{(j_1,\ldots,j_N)\in S_u} \frac{\mathbb{P}\big(\tau_1 \in (y_{1j_1}^{(1)}, y_{1j_1}^{(2)}),\ldots,\tau_N \in (y_{Nj_N}^{(1)}, y_{Nj_N}^{(2)})\big)}{\mathbb{P}\big(\tau_1 \in (y_{1u_1}^{(1)}, y_{1u_1}^{(2)}),\ldots,\tau_N \in (y_{Nu_N}^{(1)}, y_{Nu_N}^{(2)})\big)} \right\}^{-1}$$

$$= \left\{ 1 + \sum_{(j_1,\ldots,j_N)\in S_u} \lim_{\epsilon \to 0^+} \frac{\dfrac{\mathbb{P}\big(\tau_1\in(y_{1j_1}^{(1)},y_{1j_1}^{(2)}),\ldots,\tau_N\in(y_{Nj_N}^{(1)},y_{Nj_N}^{(2)})\big)}{(y_{1j_1}^{(2)}-y_{1j_1}^{(1)})\cdots(y_{Nj_N}^{(2)}-y_{Nj_N}^{(1)})}}{\dfrac{\mathbb{P}\big(\tau_1\in(y_{1u_1}^{(1)},y_{1u_1}^{(2)}),\ldots,\tau_N\in(y_{Nu_N}^{(1)},y_{Nu_N}^{(2)})\big)}{(y_{1u_1}^{(2)}-y_{Nu_N}^{(1)})\cdots(y_{Nu_N}^{(2)}-y_{Nu_N}^{(1)})}} \right.$$

$$\left. \times \prod_{i=1}^N \left(\frac{\frac{(y_{ij_i}^{(2)}-y_{ij_i}^{(1)})}{2\epsilon}}{\frac{(y_{iu_i}^{(2)}-y_{iu_i}^{(1)})}{2\epsilon}} \right) \right\}^{-1}$$

$$= \left\{ 1 + \sum_{(j_1,\ldots,j_N)\in S_u} \frac{f\big(x_{1j_1},\ldots,x_{Nj_N}\big)}{f\big(x_{1u_1},\ldots,x_{Nu_N}\big)} \prod_{i=1}^N \left| \frac{H_i'(x_{iu_i})}{H_i'(x_{ij_i})} \right| \right\}^{-1},$$

where we recall that $S_u = \{1,2\}^N \setminus \{(u_1,\ldots,u_N)\}$ and for $i = 1,\ldots,N$ and any $x > 0$,

$$H_i'(x) = \frac{((\mu_i x)^2 - (b_i - x_i)^2)}{\sigma_i^2 x^2}.$$

Let $(v_1,\ldots,v_N)$ be equal to the observation of Chi-squared vector defined in (6.1) and let $(X_{1u_1},\ldots,X_{Nu_N})$ be the root labeled $(u_1,\ldots,u_N)$ of (6.1), then the 'probability' (it is a random variable) of choosing this root is given as

$$p_{u_1\ldots u_N} = \mathbb{P}_{(u_1,\ldots,u_N)}((\chi_1^2,\ldots,\chi_N^2)).$$

It remains to generate the probability distribution $\{p_{u_1\ldots u_N}\}_{(u_1,\ldots,u_N)\in\{1,2\}^N}$. It can be simulated by generating an independent uniform random variable $U \sim Unif(0,1)$ and the following random partition of interval $[0,1]$:

$$[0,1] = \bigcup_{u_1,\ldots,u_N\in\{1,2\}} I_{u_1\ldots u_N},$$

with for almost every $\omega \in \Omega$, $\mathbb{P}(U \in I_{u_1\ldots u_N}(\omega)) = p_{u_1\ldots u_N}(\omega)$. This is equivalent to (2.6). Finally

$$\big(H_1(X_{1u_1}),\ldots,H_N(X_{Nu_N})\big) = (\chi_1^2,\ldots,\chi_N^2)$$

and

$$(\tau_1,\ldots,\tau_N) \sim \sum_{u_1,\ldots,u_N\in\{1,2\}} (X_{1u_1},\ldots,X_{Nu_N})\mathbb{1}_{I_{u_1\ldots u_N}}(U). \,\square$$

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6.2 Proof of Proposition 2.2.

Notice that we derive the idea from the seminal work by Iyengar [10]. Unfortunately, this work contains errors. Metzler [13] provided the correct formula of the two-dimensional first exit times joint density, however the one with non-zero drift with drift is not explicitly given. In this section we provide a relatively closed form of the joint density. Our work also extends the joint density of first exit times in [13] to the case where the barriers could be any real values. First we assume that for $i = 1, 2$, $x_i - b_i > 0$. The main idea is to transform the Brownian motions to independence (see [10]). By such a linear transformation, the first exit time remains invariant when the Brownian motion starts from $(x - b)$ to the barrier 0. Hence, without loss of generality, we only consider the barriers as horizontal axis $Y = 0$. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(x) = \begin{pmatrix} \sigma_1 \sqrt{1-\rho^2} & \sigma_1 \rho \\ 0 & \sigma_2 \end{pmatrix}^{-1} x,$$

be the transformation of the vector of Brownian motions $X(t) = (X_1(t), X_2(t))$ with correlation ρ and variances σ_1^2, σ_2^2 to the independent standard Brownian motions. Denote the latter by $Z(t) = (Z_1(t), Z_2(t)) = T(X(t))$ and $z_0 = Z(0) = (r_0 \cos \theta_0, r_0 \sin \theta_0)$. After transformation T, the horizontal barrier line $Y = 0$ remains the same and $Y = b_2$ turns to be the line $Y = (\tan \alpha)X$. Note that the notations (r_0, θ_0, α) are exactly the same as mentioned in the joint density of (τ_1, τ_2). The explicit form of joint density depends on the conditions $\tau_1 < \tau_2$ and $\tau_2 < \tau_1$.

1. For $0 < s < t$, this shows $\tau_1 < \tau_2$. Following the argument of Metzler [13], the joint density of (τ_1, τ_2) is given as

$$\int_{r=0}^{+\infty} \frac{\mathbb{P}(\tau_1 \in \mathrm{d}s, Z(\tau_1) \in \mathrm{d}z, z = r(\cos \alpha, \sin \alpha))}{\mathrm{d}s}$$
$$\times \frac{\mathbb{P}(\tau_2 - \tau_1 \in \mathrm{d}(t - s) | \tau_1 \in \mathrm{d}s, Z(\tau_1) \in \mathrm{d}z, z = r(\cos \alpha, \sin \alpha))}{\mathrm{d}(t - s)}$$
$$= \int_0^{+\infty} e^{\gamma'(z-z_0) - \frac{|\gamma|^2 s}{2}} \frac{\pi}{\alpha^2 sr} e^{-\frac{r^2 + r_0^2}{2s}} \sum_{n=1}^{+\infty} n \sin\left(\frac{n\pi(\alpha - \theta_0)}{\alpha}\right) I_{n\pi/\alpha}\left(\frac{rr_0}{s}\right)$$
$$\times \left(\frac{r \sin \alpha}{\sqrt{2\pi(t-s)^3}} e^{-\frac{(r \sin \alpha + \gamma_2(t-s))^2}{2(t-s)}} \right) \mathrm{d}r,$$

where

(i) $\gamma = (\gamma_1 \ \gamma_2)' = T\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \left(\frac{\sigma_2 \mu_1 - \sigma_1 \mu_2 \rho}{\sigma_1 \sigma_2 \sqrt{1-\rho^2}} \ \frac{\mu_2}{\sigma_2} \right)'$, with γ' being the transpose of γ and $|\gamma|$ being the Euclidian norm.

(ii) By easy computation, we can get α, r_0, θ_0 as in Proposition 2.2, with $\tilde{\rho} = \rho$.

(iii) $z = (r \cos \alpha, r \sin \alpha)$ is the polar coordinates of the transformation of the exit position $Z(\tau_1) = (Z_1(\tau_1), Z_2(\tau_1))$.

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(iv) By Iyengar [10] and Metzler [13],

$$\frac{\mathbb{P}(\tau_1 \in \mathrm{d}s, Z(\tau_1) \in \mathrm{d}z, z = r(\cos\alpha, \sin\alpha))}{\mathrm{d}s}$$

$$= e^{\gamma'(z-z_0) - \frac{|\gamma|^2 s}{2}} \frac{\pi}{\alpha^2 sr} e^{-\frac{r^2 + r_0^2}{2s}} \sum_{n=1}^{+\infty} n \sin\left(\frac{n\pi(\alpha - \theta_0)}{\alpha}\right) I_{n\pi/\alpha}\left(\frac{rr_0}{s}\right)$$

is the density of time for $Z(t)$ to first exit from the line $Y = (\tan\alpha)X$ and

$$\frac{\mathbb{P}(\tau_2 - \tau_1 \in \mathrm{d}(t-s)|\tau_1 \in \mathrm{d}s, Z(\tau_1) \in \mathrm{d}z, z = r(\cos\alpha, \sin\alpha))}{\mathrm{d}(t-s)}$$

$$= \frac{r\sin\alpha}{\sqrt{2\pi(t-s)^3}} e^{-\frac{(r\sin\alpha + \gamma_2(t-s))^2}{2(t-s)}} \, \mathrm{d}r$$

is in fact the inverse Gaussian density on $t-s$ which denotes the remaining time for $Z_2(t)$ to first exit its barrier, once $Z_1(t)$ exits. By martingale property, this event can be regarded as the first exit time of a standard Brownian motion with drift γ_2, starting from $r\sin\alpha$, to the barrier 0 (see Fig. 1 for $\tau_1 < \tau_2$).

2. For $0 < t < s$ (this shows $\tau_2 < \tau_1$), the joint density can be obtained in a similar way:

$$\frac{\mathbb{P}(\tau_2 \in \mathrm{d}t, Z(\tau_2) \in \mathrm{d}z, z = (r, 0))}{\mathrm{d}t}$$

$$= \int_{r=0}^{+\infty} \frac{\mathbb{P}(\tau_2 \in \mathrm{d}t, Z(\tau_2) \in \mathrm{d}z, z = (r, 0))}{\mathrm{d}t}$$

$$\times \frac{\mathbb{P}(\tau_1 - \tau_2 \in \mathrm{d}(s-t)|\tau_2 \in \mathrm{d}t, Z(\tau_2) \in \mathrm{d}z, z = (r, 0))}{\mathrm{d}(s-t)}$$

$$= \int_0^{+\infty} e^{\gamma'(z-z_0) - \frac{|\gamma|^2 t}{2}} \frac{\pi}{\alpha^2 tr} e^{-\frac{r^2 + r_0^2}{2t}} \sum_{n=1}^{+\infty} n \sin\left(\frac{n\pi\theta_0}{\alpha}\right) I_{n\pi/\alpha}\left(\frac{rr_0}{t}\right)$$

$$\times \left(\frac{r\sin\alpha}{\sqrt{2\pi(s-t)^3}} e^{-\frac{(r\sin\alpha + (s-t)(\gamma_1 \cos(\alpha - \pi/2) + \gamma_2 \cos(\pi - \alpha)))^2}{2(s-t)}}\right) \mathrm{d}r,$$

where we just need to remark that $z = (r, 0)$ and the latter inverse Gaussian density (on $s-t$) denotes the remaining time for $Z_1(t)$ to first exit its barrier, once $Z_2(t)$ exits. It can be further regarded as the first exit time of a standard Brownian motion with drift $\gamma_1 \cos(\alpha - \pi/2) + \gamma_2 \cos(\pi - \alpha))$, starting from $r\sin\alpha$, until the barrier 0.

Now it suffices to simplify the above two formulae. For example, let us simplify the first one: when $0 < s < t$,

$$f(s,t) = \int_0^{+\infty} e^{\gamma_1(r\cos\alpha - r_0\cos\theta_0) + \gamma_2(r\sin\alpha - r_0\sin\theta_0) - \frac{(\gamma_1^2+\gamma_2^2)s}{2}}$$

$$\times \frac{\pi}{\alpha^2 sr} e^{-\frac{r^2+r_0^2}{2s}} \sum_{n=1}^{+\infty} n\sin\left(\frac{n\pi(\alpha-\theta_0)}{\alpha}\right) I_{n\pi/\alpha}\left(\frac{rr_0}{s}\right)$$

$$\times \frac{r\sin\alpha}{\sqrt{2\pi(t-s)^3}} e^{-\frac{(r\sin\alpha+\gamma_2(t-s))^2}{2(t-s)}}\, \mathrm{d}r$$

$$= \frac{\pi\sin\alpha}{\alpha^2 s\sqrt{2\pi(t-s)^3}} e^{-\gamma_1 r_0\cos\theta_0 - \gamma_2 r_0\sin\theta_0 - \frac{r_0^2}{2s} - \frac{\gamma_1^2 s+\gamma_2^2 t}{2}}$$

$$\times \int_0^{+\infty} \sum_{n=1}^{+\infty} n\sin\left(\frac{n\pi(\alpha-\theta_0)}{\alpha}\right) e^{r\gamma_1\cos\alpha - r^2\left(\frac{1}{2s}+\frac{s\sin^2\alpha}{2s(t-s)}\right)} I_{n\pi/\alpha}\left(\frac{rr_0}{s}\right)\, \mathrm{d}r$$

$$= \sqrt{\frac{\pi}{2}}\frac{\sin\alpha}{\alpha^2 s\sqrt{(t-s)^3}} e^{-r_0\left(\frac{r_0}{2s}+\gamma_1\cos\theta_0+\gamma_2\sin\theta_0\right) - \frac{\gamma_1^2 s+\gamma_2^2 t}{2}}$$

$$\times \sum_{n=1}^{+\infty} n\sin\left(\frac{n\pi(\alpha-\theta_0)}{\alpha}\right) \int_0^{+\infty} e^{r\gamma_1\cos\alpha - r^2\left(\frac{t-s\cos^2\alpha}{2s(t-s)}\right)} I_{n\pi/\alpha}\left(\frac{rr_0}{s}\right)\, \mathrm{d}r.$$

When $0 < t < s$, we can pursue the same approach to simplify the expression (see Fig. 1 for $\tau_1 > \tau_2$). Finally, we get

1. For $0 < s < t$,

$$f(s,t) = \sqrt{\frac{\pi}{2}}\frac{\sin\alpha}{\alpha^2 s\sqrt{(t-s)^3}} e^{-r_0\left(\frac{r_0}{2s}+\gamma_1\cos\theta_0+\gamma_2\sin\theta_0\right) - \frac{\gamma_1^2 s+\gamma_2^2 t}{2}}$$

$$\times \sum_{n=1}^{+\infty} n\sin\left(\frac{n\pi(\alpha-\theta_0)}{\alpha}\right) \int_0^{+\infty} e^{\gamma_1\cos\alpha - r^2\left(\frac{t-s\cos^2\alpha}{2s(t-s)}\right)} I_{n\pi/\alpha}\left(\frac{rr_0}{s}\right)\, \mathrm{d}r. \tag{6.3}$$

2. For $0 < t < s$,

$$f(s,t) = \sqrt{\frac{\pi}{2}}\frac{\sin\alpha}{\alpha^2 t\sqrt{(s-t)^3}} \exp\left(-r_0\left(\frac{r_0}{2t}+\gamma_1\cos\theta_0+\gamma_2\sin\theta_0\right)\right.$$

$$\left. -\frac{(\gamma_1^2+\gamma_2^2)t}{2} - \frac{(\gamma_1\sin\alpha-\gamma_2\cos\alpha)^2(s-t)}{2}\right) \sum_{n=1}^{+\infty} n\sin\left(\frac{n\pi\theta_0}{\alpha}\right)$$

$$\times \int_0^{+\infty} e^{-r\left(\gamma_1\cos^2\alpha+\gamma_2\sin\alpha\cos\alpha\right) - r^2\left(\frac{s-t\cos^2\alpha}{2t(s-t)}\right)} I_{n\pi/\alpha}\left(\frac{rr_0}{t}\right)\, \mathrm{d}r. \tag{6.4}$$
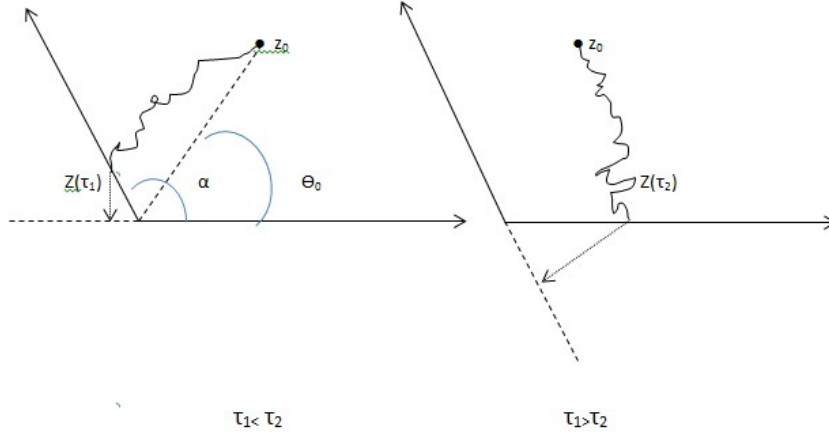
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Figure 1: An example of the trajectories of $\{Z(t)\}_{t\geq 0}$ for $\tau_1 < \tau_2$ and $\tau_1 > \tau_2$

Now we are going to derive the most generalized density, where the barriers could be any real values. To this end we first observe that, when $x_1 - b_1 > 0$, $x_2 - b_2 > 0$, there exists a function $\tilde{f}$ such that

$$f(s,t) = \tilde{f}\big(x_1 - b_1, x_2 - b_2, \mu_1, \mu_2, \sigma_1, \sigma_2, \rho, s, t\big)$$

By using the symmetric property of Brownian motion, when $x_i - b_i < 0$ for $i = 1$ or 2, we notice that

$$\begin{aligned}
\tau_i &= \inf\big\{t > 0 : X_i(t) = b_i,\ X_i(0) = x_i\big\} \\
&= \inf\big\{t > 0 : -X_i(t) = -b_i, -X_i(0) = -x_i\big\},
\end{aligned}$$

where $\{-X_i(t)\}_{t\geq 0}$ is a Brownian motion with drift $-\mu_i = (sgn(x_i - b_i))\mu_i$ starting from $-x_i$. Therefore the first exit times of $(X_1(t), X_2(t))$ with drifts (μ_1, μ_2), starting from (x_1, x_2) to the barriers (b_1, b_2) are almost surely equal to the first exit times of $\big((sgn(x_1 - b_1))X_1(t), (sgn(x_2 - b_2))X_2(t)\big)$, starting from the initial values $\big((sgn(x_1 - b_1))x_1, (sgn(x_2 - b_2))x_2\big)$ to the barriers $\big((sgn(x_1 - b_1))b_1, (sgn(x_2 - b_2))b_2\big)$. Remark that

$$Corr\big((sgn(x_1 - b_1))X_1(t), (sgn(x_2 - b_2))X_2(t)\big) = \frac{sgn(b_1 - x_1)}{sgn(b_2 - x_2)}\rho = \tilde{\rho}.$$

Finally, we have for all $x_i \neq b_i$, $i = 1, 2$,

$$f(s,t) = \tilde{f}\big(|b_1 - x_1|, |b_2 - x_2|, (sgn(x_1 - b_1))\mu_1, (sgn(x_2 - b_2))\mu_2, \sigma_1, \sigma_2, \tilde{\rho}, s, t\big).$$

For any $x_i \neq b_i$, $i = 1, 2$, plugging the arguments

$$\big(|b_1 - x_1|, |b_2 - x_2|, (sgn(x_1 - b_1))\mu_1, (sgn(x_2 - b_2))\mu_2, \sigma_1, \sigma_2, \tilde{\rho}\big)$$

into (6.3) and (6.4), we get Proposition 2.2. $\square$

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6.3 Proof of Corollary 2.1.

To prove Corollary 2.1, we rely on Lemma 2.1 and the following lemma which is known as a consequence of Sklar's Theorem:

Lemma 6.1. *Let F_{χ^2} be the cumulative probability distribution function of $\chi^2 \sim \chi^2(1)$, then $U = F_{\chi^2}(\chi^2) \sim Unif(0,1)$.*

Now we are ready to prove Corollary 2.1. Since

$$(\chi_1^2, \chi_2^2) = \Big(\frac{(b_1 - x_1)^2}{\sigma_1^2 \tau_1}, \frac{(b_2 - x_2)^2}{\sigma_2^2 \tau_2} \Big),$$

then by Lemma 6.1,

$$(U_1, U_2) := \Big(F_{\chi^2} \big(\frac{(b_1 - x_1)^2}{\sigma_1^2 \tau_1} \big), F_{\chi^2} \big(\frac{(b_2 - x_2)^2}{\sigma_2^2 \tau_2} \big) \Big)$$

has marginal distribution $Unif(0,1)$. Observe that for any $x \geq 0$, $F_{\chi^2}(x) = \mathbb{P}(Z^2 \leq x) = 2\varphi(\sqrt{x}) - 1$. Then the correlation of (U_1, U_2) can be given by using the joint density of (τ_1, τ_2):

$$Corr(U_1, U_2) = Corr\big(F_{\chi^2} \big(\frac{(b_1 - x_1)^2}{\sigma_1^2} \big), F_{\chi^2} \big(\frac{(b_2 - x_2)^2}{\sigma_2^2} \big) \big)$$

$$= Corr\big(2\varphi(\frac{|b_i - x_i|}{\sigma_i \sqrt{\tau_i}}) - 1, 2\varphi(\frac{|b_j - x_j|}{\sigma_j \sqrt{\tau_j}}) - 1 \big)$$

$$= \frac{\mathbb{E}\big(\big(2\varphi(\frac{|b_i - x_i|}{\sigma_i \sqrt{\tau_i}}) - 1 \big) \big(2\varphi(\frac{|b_j - x_j|}{\sigma_j \sqrt{\tau_j}}) - 1 \big) \big) - \mathbb{E}(U_1)\mathbb{E}(U_2)}{\sqrt{Var(U_1)Var(U_2)}}$$

$$= 12 \Big(\int_0^{+\infty} \int_0^{+\infty} \big(2\varphi(\frac{|b_i - x_i|}{\sigma_i \sqrt{s}}) - 1 \big) \big(2\varphi(\frac{|b_j - x_j|}{\sigma_j \sqrt{t}}) - 1 \big) f_{ij}(s, t) \, ds \, dt - \frac{1}{4} \Big).$$

Therefore by Lemma 2.1, there exists a Gaussian vector such that (2.1) holds. By the fact that for $i = 1, 2$,

$$\chi_i^2 = F_{\chi^2}^{-1}(U_i) \sim F_{\chi^2}^{-1}(\varphi(Z_i)),$$

and by observing that $F_{\chi^2}^{-1}(x) = \big(\varphi^{-1}(\frac{x+1}{2}) \big)^2$, Corollary 2.1 holds. □

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